When a complete solution is received, it will be published.

- **5376:** *Proposed by Arkady Alt , San Jose ,CA*

Let $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n$ be positive real numbers such that
$b_1 < a_1 < b_2 < a_2 < ... < a_{n-1} < b_n < a_n$.
Let

$$F(x) = \frac{(x - b_1)(x - b_2) \dots (x - b_n)}{(x - a_1)(x - a_2) \dots (x - a_n)}.$$

Prove that $F'(x) < 0$ for any $x \in Dom(F)$.

Solution 1 by Albert Stadler, Herrliberg, Switzerland

We note that $F(x) = \prod_{m=1}^{n} \frac{x - b_m}{x - a_m}$ is a rational function with simple poles at $x = a_m$, $1 \leq m \leq n$.

The residue of $F(x)$ at $x = a_\mu$ equals $(a_\mu - b_\mu) \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m} > 0$, since $a_\mu > b_\mu$ and

$\frac{a_\mu - b_m}{a_\mu - a_m} > 0$, for $m \neq \mu$.

So $f(x) = F(x) - \sum_{\mu=1}^{n} \frac{a_\mu - b_\mu}{x - a_\mu} \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m}$ is a bounded entire function which implies that $f(x)$ is a constant. We conclude $f'(x) = 0$ which implies

$F'(x) = -\sum_{\mu=1}^{n} \frac{a_\mu - b_\mu}{(x - a_\mu)^2} \prod_{m \neq \mu} \frac{a_\mu - b_m}{a_\mu - a_m} < 0$ for any $x \in$Dom (F).

Solution 2 by Ethan Gegner (student), Taylor University, Upland, IN

For all $x \in Dom(F)$, we have

$$F'(x) = \frac{\left(\prod_{i=1}^{n}(x - a_i)\right)\left(\prod_{i=1}^{n}(x - b_i)\right)' - \left(\prod_{i=1}^{n}(x - b_i)\right)\left(\prod_{i=1}^{n}(x - a_i)\right)'}{\left(\prod_{i=1}^{n}(x - a_i)\right)^2} \qquad (1)$$

Suppose $x = b_j$ for some $1 \leq j \leq n$. Then

$$F'(x) = \frac{\prod_{i \neq j}(x - b_i)}{\prod_{i=1}^{n}(x - a_i)} = \frac{1}{(x - a_j)} \prod_{i \neq j} \frac{x - b_i}{x - a_i} < 0$$

since $x = b_j < a_j$ and $\frac{x - b_i}{x - a_i} > 0$ for all $i \neq j$.

Now suppose $x \notin \{b_1, \dots, b_n\}$. Then $F(x) \neq 0$, so by equation (1) we have

$$\frac{F'(x)}{F(x)} = \frac{\left(\prod_{i=1}^{n}(x - b_i)\right)'}{\prod_{i=1}^{n}(x - b_i)} - \frac{\left(\prod_{i=1}^{n}(x - a_i)\right)'}{\prod_{i=1}^{n}(x - a_i)} = \sum_{i=1}^{n} \left(\frac{1}{x - b_i} - \frac{1}{x - a_i}\right)$$

$$= \sum_{i=1}^{n} \frac{b_i - a_i}{(x - b_i)(x - a_i)} \qquad (2)$$

If $x < b_1$ or $x > a_n$, then $F(x) > 0$, and $\frac{b_i - a_i}{(x - b_i)(x - a_i)} < 0$ for all $1 \leq i \leq n$, whence $F'(x) < 0$. Suppose there exists some $1 \leq j \leq n - 1$ such that $a_j < x < b_{j+1}$. Then for every

$1 \le i \le n$, $x - b_i$ and $x - a_i$ have the same sign, whence $\dfrac{b_i - a_i}{(x - b_i)(x - a_i)} < 0$ and $F(x) = \prod_{i=1}^{n} \dfrac{x - b_i}{x - a_i} > 0$. Thus, equation (2) implies $F'(x) < 0$.

Finally, suppose that $b_j < x < a_j$ for some $1 \le j \le n$. Then

$$\frac{F'(x)}{F(x)} = \sum_{i=1}^{n} \left(\frac{1}{x - b_i} - \frac{1}{x - a_i} \right) = \frac{1}{x - b_1} - \frac{1}{x - a_n} + \sum_{i=1}^{n-1} \left(\frac{1}{x - b_{i+1}} - \frac{1}{x - a_i} \right) > 0$$

since every term on the right hand side is positive. Moreover, $F(x) = \dfrac{x - b_j}{x - a_j} \prod_{i \ne j} \dfrac{x - b_i}{x - a_i} < 0$, so again $F'(x) < 0$.

Solution 3 by the proposer

Lemma.

$F(x)$ can be represented in form

$$F(x) = 1 + \sum_{k=1}^{n} \frac{c_k}{x - a_k},$$

where $c_k, k = 1, 2, ..., n$ are some positive real numbers.

Proof.

Let $F_k(x) := \dfrac{(x - b_1)(x - b_2)...(x - b_k)}{(x - a_1)(x - a_3)...(x - a_k)}$, $k \le n$.

We will prove by Math Induction that for any $k \le n$ there are positive numbers $c_k(i), i = 1, ..., k$ such that $F_k(x) = 1 + \sum\limits_{i=1}^{k} \dfrac{c_k(i)}{x - a_i}$.

Let $d_k := a_k - b_k > 0, k = 1, 2, ..., n$.

Note that $F_1(x) = \dfrac{x - b_1}{x - a_1} = \dfrac{x - a_1 + a_1 - b_1}{x - a_1} = 1 + \dfrac{d_1}{x - a_1}$.

Since $\dfrac{x - b_{k+1}}{x - a_{k+1}} = 1 + \dfrac{d_{k+1}}{x - a_{k+1}}$ then in supposition $F_k(x) = 1 + \sum\limits_{i=1}^{k} \dfrac{c_k(i)}{x - a_i}$, where $c_k(i) > 0, i = 1, ..., k < n$ we obtain

$$\begin{aligned}
F_{k+1}(x) &= F_k(x) \cdot \frac{x - b_{k+1}}{x - a_{k+1}} = \left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{x - a_i}\right)\left(1 + \frac{d_{k+1}}{x - a_{k+1}}\right) \\
&= 1 + \frac{d_{k+1}}{x - a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x - a_i} + \sum_{i=1}^{k} \frac{d_{k+1} c_k(i)}{(x - a_i)(x - a_{k+1})} \\
&= 1 + \frac{d_{k+1}}{x - a_{k+1}} + \sum_{i=1}^{k} \frac{c_k(i)}{x - a_i} - \sum_{i=1}^{k} \frac{d_{k+1} c_k(i)}{a_{k+1} - a_i} \left(\frac{1}{x - a_i} - \frac{1}{x - a_{k+1}} \right) \\
&= 1 + \frac{d_{k+1}}{x - a_{k+1}} \left(1 + \sum_{i=1}^{k} \frac{c_k(i)}{a_{k+1} - a_i} \right) + \sum_{i=1}^{k} \frac{c_k(i)}{x - a_i} \left(1 - \frac{d_{k+1}}{a_{k+1} - a_i} \right)
\end{aligned}$$

$$= \quad 1+\frac{d_{k+1}F_k\left(a_{k+1}\right)}{x-a_{k+1}}+\sum_{i=1}^{k}\frac{c_k\left(i\right)}{x-a_i}\cdot\frac{b_{k+1}-a_i}{a_{k+1}-a_i}.$$

Since $F_k\left(a_{k+1}\right)>0$ and $b_{k+1}-a_i=\left(b_{k+1}-a_k\right)+\left(a_k-a_i\right)>0$ then

$c_{k+1}\left(k+1\right)=d_{k+1}F_k\left(a_{k+1}\right)>0,\ c_{k+1}\left(i\right):=\dfrac{\left(b_{k+1}-a_i\right)c_k\left(i\right)}{a_{k+1}-a_i}>0, i=1,2,...,k$

and $F_{k+1}\left(x\right)=1+\sum\limits_{i=1}^{k+1}\dfrac{c_{k+1}\left(i\right)}{x-a_i}$.

Therefore, since $F\left(x\right)=1+\sum\limits_{k=1}^{n}\dfrac{c_k}{x-a_k}$ and $c_k>0, k=1,2,...,n$ then
$F'\left(x\right)=-\sum\limits_{k=1}^{n}\dfrac{c_k}{\left(x-a_k\right)^2}<0$ for any $x\in Dom\left(F\right)=\left\{a_1,a_2,...,a_n\right\}.$

Solution 4 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

We find $F'(x)$,

$$\begin{aligned} F'(x) \quad = \quad & \frac{b_1-a_1}{(x-a_1)^2}\cdot\frac{x-b_2}{x-a_2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{x-b_n}{x-a_n}+\frac{x-b_1}{x-a_1}\cdot\frac{b_2-a_2}{(x-a_2)^2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{x-b_n}{x-a_n}+\ldots+\\ & \frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdots\frac{x-b_{j-1}}{x-a_{j-1}}\cdot\frac{b_j-a_j}{(x-a_j)^2}\cdot\frac{x-b_{j+1}}{x-a_{j+1}}\cdots\cdot\frac{x-b_n}{x-a_n}+\ldots+\\ & \frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{b_n-a_n}{(x-a_n)^2}. \qquad (1)\end{aligned}$$

We set

$$\begin{aligned} D_1(x) &= \frac{b_1-a_1}{(x-a_1)^2}\cdot\frac{x-b_2}{(x-a_2)}\cdot\frac{x-b_3}{(x-a_3)}\cdots\frac{x-b_n}{x-a_n}\\ D_2(x) &= \frac{x-b_1}{x-a_1}\cdot\frac{b_2-a_2}{(x-a_2)^2}\cdot\frac{x-b_3}{(x-a_3)}\cdots\frac{x-b_n}{x-a_n}\\ &\vdots\\ D_j(x) &= \frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdots\frac{x-b_{j-1}}{x-a_{j-1}}\cdot\frac{b_j-a_j}{(x-a_j)^2}\cdot\frac{x-b_{j+1}}{x-a_{j+1}}\cdots\frac{x-b_n}{x-a_n}\\ &\vdots\\ D_n(x) &= \frac{x-b_1}{x-a_1}\cdot\frac{x-b_2}{x-a_2}\cdot\frac{x-b_3}{x-a_3}\cdots\frac{b_n-a_n}{(x-a_n)^2}.\ \text{ Then}\end{aligned}$$

$$F'(x)=\sum_{k=1}^{n}D_k(x). \qquad (2)$$

We note that because

$$0<b_1<a_1<b_2<a_2<\cdots<a_{n-1}<b_n<a_n \qquad (3)$$

we have

$$\frac{b_j - a_j}{(x-a_j)^2} < 0, \text{for all } j \text{ with } 1 \le j \le n. \quad (4)$$

Let $x \in Dom(F)$, then we consider the following cases:

Case 1. Let $x = b_{j_0}$, for some $j_0 \in \{1, 2, \cdots, n\}$, then $D_j(b_{j_0}) = 0$, for all $j \neq j_0$, and because of (3), $\dfrac{b_{j_0} - b_j}{b_{j0} - a_j} > 0$, for all $j \neq j_0$ and with (4) we conclude that $F'(b_{j_0}) < 0$.

Case 2. Let $x < b_1$, then for all j with $1 \le j \le n$, and by using (3), we conclude that and that $\dfrac{x - b_j}{x - a_j} > 0. \quad (5)$
And then by (4) and (5) we get equation (6) that $D_j(x < b_1) < 0$, for all j with $1 \le j \le n$, and this implies that $F'(x < b_1) < 0$.

Case 3. Let $x \in (b_{j_0}, a_{j_0})$ for some $j_0 \in \{1, 2, \cdots, n\}$, we will show that $F(x)$ is decreasing on (b_{j_0}, a_{j_0}). We know that by (4) and (3),, each function $f_j(x) = \dfrac{x - b_j}{x - a_j}$ is decreasing and positive on (b_{j_0}, a_{j_0}), when $j \neq j_0$, then for all $s, t \in (b_{j_0}, a_{j_0})$ with $s < t$ we have

$$f_j(t) > f_j(s), \quad (7)$$

also $f_{j_0}(x) = \dfrac{x - b_{j_0}}{x - a_{j_0}}$ is decreasing but negative on (b_{j_0}, a_{j_0}) and

$$f_{j_0}(t) > f_{j_0}(s). \quad (8)$$

Now using (7) and (8), we have $\prod_{j=1}^{n} f_j(t) > \prod_{j=1}^{n} f_j(s)$, that is $F(t) > F(s)$, whenever $s, t \in (b_{j_0}, a_{j_0})$ with $s < t$, the means $F(x)$ is decreasing on (b_{j_0}, a_{j_0}) or $F(x) < 0$ on (b_{j_0}, a_{j0}).

Case 4. Let $x \in \left(a_{j_0}, b_{j_{0+1}}\right)$, for some $j_0 \in \{1, 2, \cdots, n-1\}$, then $f_j(x) = \dfrac{x - b_j}{x - a_j} > 0$, on $\left(a_{j_0}, b_{j_{0+1}}\right)$, for $j \in \{1, 2, \cdots, n\}$, and by (4) and (2), we conclude that $F'(x) < 0$, on $\left(a_n, b_{j_{0+1}}\right)$.

Case 5. Let $x \in (b_n, \infty)$, then $f_j(x) = \dfrac{x - b_j}{x - a_j} > 0$, on (b_n, ∞) for all $j \in \{1, 2, \cdots, n\}$, and by (4) and (2), we conclude that $F'(x) < 0$, on (b_n, ∞).
Combining the results of Cases 1-5, we conclude that $F'(x) < 0$ for any $x \in Dom(F)$.

Also solved by Ed Gray, Highland Beach, FL; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy, and Toshihiro Shimizu, Kawasaki, Japan.

- **5377:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Show that if A, B, C are the measures of the angles of any triangle ABC and a, b, c the measures of the length of its sides, then holds

$$\prod_{cyclic} \sin^{1/3}(|A - B|) \le \sum_{cyclic} \frac{a^2 + b^2}{3ab} \sin(|A - B|).$$

Solution 1 by Andrea Fanchini Cantú, Italy